Yau College Math Competition 2024
Final Probability and Statistics
Team Problems (June 8-9, 2024)

Choose at least 2 of the following 3 problems.

Problem 1. Let X_n, $n = 1, 2, \cdots$, be independent uniform $[0, 1]$ random variables and $X_0 = x \in [0, 1]$ be a constant. Define

$$N = \min\{n : X_n < X_{n-1}\}.$$

Find at least two different methods to evaluate $\mathbb{E}(N)$.

Problem 2. Assume $Y_1, \ldots, Y_n$ iid from a distribution with density function $f(y \mid \alpha, \beta)$. Without knowing α, β, we can obtain the MLE $\hat{\alpha}, \hat{\beta}$. If we know the true $\alpha = \alpha_0$, then we can obtain the MLE $\tilde{\beta}$.

(1) Show that the asymptotic variance of $\hat{\beta}$ is larger than or equal to that of $\tilde{\beta}$.

(2) Find an example that the asymptotic variance of $\hat{\beta}$ is strictly larger than that of $\tilde{\beta}$.

(3) Find an example that the asymptotic variance of $\hat{\beta}$ is equal to that of $\tilde{\beta}$.

Problem 3. Let $A = (a_{ij})_{m,n}$ be an m-by-n matrix with iid $N(0, 1)$ entries. We assume that $m \leq n$ and denote the singular values of A by $s_1 \geq \ldots \geq s_m$ (by definition s_i^2's are eigenvalues of $AA^\top$). We also have the variational characterizations

$$s_1 = \max_{u \in S^{m-1}, v \in S^{n-1}} u^\top A v, \qquad s_m = \min_{u \in S^{m-1}} \max_{v \in S^{n-1}} u^\top A v.$$

Provide an upper bound of $\mathbb{E}s_1$ and a lower bound of $\mathbb{E}s_m$.